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Abstract

In 1687 Isaac Newton published PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA, where the classical analytic dynamics was formulated. But Newton formulated also a discrete version, which is the central difference algorithm, known as the Verlet algorithm and which is used in computer simulations. Here we show, that the discrete dynamics for Kepler's equation has the same solutions as the analytic. The discrete positions of a celestial body are on an ellipse, which is the exact solution for a shadow Hamiltonian. T. D Lee and others have suggested, that difference equations are more fundamental, and differential equations are regarded as approximation. If so, Isaac Newton will also be the founder of the discrete dynamics.

I. INTRODUCTION

Newton formulated the dynamics of an object by means of a differential equation, and in the Lagrange-Hamilton formulation of the classical dynamics the position $\mathbf{r}(t)$ and momentum $\mathbf{p}(t)$ are analytic dynamical variables of a coherent time. But in 1983 T. D. Lee wrote a paper [1] entitled, "Can Time Be a Discrete Dynamical Variable?"; which led to a series of publications by Lee and collaborators on the formulation of fundamental dynamics in terms of difference equations, but with exact invariance under continuous groups of translational and rotational transformations. Lee's analysis covers not only classical mechanics [1], but also non relativistic quantum mechanics and relativistic quantum field theory [2], and Gauge theory and Lattice Gravity [3].

Today almost all numerical integrations of classical dynamics are performed by discrete dynamics, by updating the positions and momenta at discrete times. The classical discrete dynamics is the classical limit dynamics of a general discrete dynamics. The fundamental length and time in quantum dynamics are the Planck length $l_{\mathrm{P}} \approx 1.6 \times 10^{-35}$m and Planck time $t_{\mathrm{P}} \approx 5.4 \times 10^{-44}$ s [4], and they are immensely smaller than the length unit (given by the floating point precision) and time increment used in the classical discrete dynamics. But the dynamics of a celestial body and the stability of galaxies is, however, determined by classical dynamics, e.g . by modifying the analytic Newtonian dynamics [5]. Here we show that Newton in fact also is the founder of the discrete classical mechanics, and that the stability of a celestial body is equal well described by the exact discrete classical mechanics as it is by the analytic dynamics.

II. NEWTONS DISCRETE DYNAMICS

In 1687 Isaac Newton published PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA. (*Principia*) [6], where he formulated the equations for the classical analytic dynamics of objects. Newton's three laws relate an object with mass m at the position, $\mathbf{r}$(t), momentum, $\mathbf{p}$(t), at time t with the force $\mathbf{F(r)}$. The English translation [7] of the Latin formulation of Newton's second law second law is

The alteration of motion(momentum) *is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*, i.e.

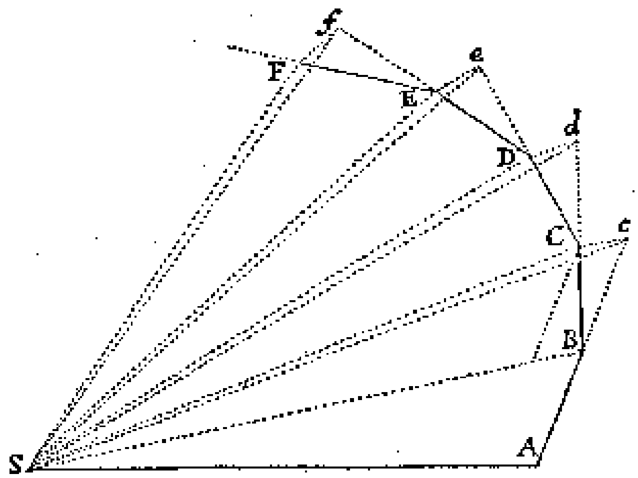

FIG. 1. Newton's figure in Principia, at his formulations of the discrete dynamics. The discrete positions are A: $\mathbf{r_A}(t_0)$; B: $\mathbf{r}_B(t_0+\delta t)$; C: $\mathbf{r}_C(t_0+2\delta t)$, etc.. The deviation from the straight line ABc (Newton's first law) is caused by a force from the position S at time $t_0+\delta t$.

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{dp}}{\mathbf{dt}}, \tag{1}$$

and in *Section II*, Newton derived an interesting relation:

PROPOSITION I. THEOREM I. The areas, which resolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described.

Newton noticed, that (see Figure 1): *For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line AB. In the second part of that time, the same would (by Law I.), if not hindered, proceed directly to c, along the line Bc equal to AB; so that by the radii AS, BS, cS, drawn to the centre, equal areas ASB, BSc, would be described. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and, turning aside the body from the right line Bc, compels it afterwards to continue its motion along the right line BC. Draw cC parallel to BS meeting BC in C; and at the end of the second part of the time, the body (by Cor. I. of the Laws) will be found in C, in the same plane with triangle ASB Join SC, and, because SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc, and therefore also to the triangle SAB.*

So according to Newton's $PROPORSITION$ the particle moves with constant momentum, $m(\mathbf{r}_B(t_0+\delta t)-\mathbf{r}_A(t_0))/\delta t$ from the position $\mathbf{r}_A(t_0)$ to the position $\mathbf{r}_B(t_0+\delta t)$ in the time interval $t \in [t_0, t_0+\delta t]$, where a force, $\mathbf{F}(\mathbf{r}_B)$ instantaneously changes the momentum. This formulation of the discrete updating of positions: $\mathbf{r}_A(t_0)$, $\mathbf{r}_B(t_0+\delta t)$ $\mathbf{r}_C(t_0+2\delta t)$,.. with constant momentum in the time intervals between the updating is the central difference algorithm

$$m\frac{\mathbf{r}(t_n+\delta t)-\mathbf{r}(t_n)}{\delta t} = m\frac{\mathbf{r}(t_n)-\mathbf{r}(t_n-\delta t)}{\delta t}r + \delta t\mathbf{F}(t_n). \tag{2}$$

The algorithm determines the $n+1$'th position from the two previous positions by

$$\mathbf{r}(t_n+\delta t) = 2\mathbf{r}(t_n) - \mathbf{r}(t_n-\delta t) + \frac{\delta t^2}{m}\mathbf{F}(t_n). \tag{3}$$

and this formulation of Newton's central difference algorithm is the so called "Verlet" algorithm [8, 9], which is used in Molecular Dynamics simulations [10, 11]. The algorithm can be reformulated, if one updates the positions in two steps with $\mathbf{v}(t_n+\delta t/2) \equiv (\mathbf{r}(t_n+\delta t)-\mathbf{r}(t_n))/\delta t$:

$$\begin{aligned}
\mathbf{v}(t_n+\delta t/2) &= \mathbf{v}(t_n-\delta t/2) + \frac{\delta t}{m}\mathbf{F}(t_n) \\
\mathbf{r}(t_n+\delta t) &= \mathbf{r}(t_n) + \delta t\mathbf{v}(t_n+\delta t/2),
\end{aligned} \tag{4}$$

and this reformulation is named the "leap-frog" algorithm. It is the discrete version of Euler's equations for Newton's analytic dynamics [12]. There exists several other reformulations of the central difference algorithm [10, 11].

Newton's classical analytic dynamics is the limit dynamics in quantum electrodynamics (QED) for relative slow motions of heavy objects. The discrete dynamics, obtained by the central difference algorithm has the same qualitative behaviour as the analytic. Is time reversible, symplectic [13], and has the same invariances: total momentum, angular momentum and energy [14] , as the analytic dynamics. Here we argue, that Newton's formulation of the discrete dynamics will be the corresponding limit dynamics, if time, forces and space are discrete [1]. The conclusion is made after simulating Kepler's planet system by Newton's central difference algorithm.

Before the formulation of the discrete dynamics for a celestial body is presented, the solution of Keplers equation for analytic dynamics is summarized in the next section.

III. THE SOLUTION OF KEPLER'S EQUATION

A. The analytic solution of Kepler's equation

Newton solved in *Principia*, Kepler's equation for the orbit of a planet. The solution of Kepler's equation [15]

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{gMm}{r(t)^2}\hat{\mathbf{r}} \tag{5}$$

for a planet with the gravitational constant g and mass m at the position $\mathbf{r}(t)$ from the Sun with mass M at the origin relates the constant energy,

$$E = 1/2m\mathbf{v}(t)^2 - gMm/r(t), \tag{6}$$

with the semi major axis in an ellipse

$$a = -gMm/2E. \tag{7}$$

The longest distance r_{max} (aphelion) from the Sun is

$$r_{max} = 2a - r_p, \tag{8}$$

where r_p is the shortest distance (perihelion) to the Sun. The eccentricity, ϵ, is

$$\epsilon = \frac{r_{max} - r_p}{r_{max} + r_p} = 1 - \frac{r_p}{a}. \tag{9}$$

and the semi minor axis, b is

$$b = a\sqrt{1-\epsilon^2}. \tag{10}$$

With the major axis in the x-direction the planet moves in a stable elliptic orbit

$$\frac{(x(t) - (a - r_p))^2}{a^2} + \frac{y(t)^2}{b^2} = 1, \tag{11}$$

for

$$0 \leq \epsilon < 1, \tag{12}$$

within a orbit period

$$T(orbit) = 2\pi\sqrt{a^3/gM}. \tag{13}$$

The velocity at perihelion, $\mathbf{v}_p(t)) = (0, vy_p)$, is in the y-direction and the energy is

$$E = 1/2mvy_p^2 - gMm/r_p, \tag{14}$$

and since $1/a = -2E/gMm = -mvy_p(t)^2 + 2/r_p$, the limit values for elliptic orbits can be expressed by the maximum velocity as

$$\sqrt{gM/r_p} \leq vy_p < \sqrt{2gM/r_p}. \tag{15}$$

Let the planet at time $t_0 = 0$ be in the perihelion of the elliptic orbit with the maximum velocity $\mathbf{v}_p = (0, vy_p)$ at the shortest distance, $\mathbf{r}_{min} = (x(t_0), y(t_0)) = (-r_p, 0)$, from the Sun, which is located at the origin. The classical orbit of a planet can be obtained from these four parameter: gM, m, r_p, vy_p (or: gM, m, r_{max}, vy_{min} at aphelion).

B. Kepler's orbit obtained by Newton's discrete central difference algorithm

The discrete dynamics can be obtained from the same parameters, gM, m, r_p, vy_p together with the discrete time increment δt. Newton's discrete dynamics for the $n+1$'th change of position of a planet is

$$m\frac{\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)}{\delta t} = m\frac{\mathbf{r}(t_n) - \mathbf{r}(t_{n-1})}{\delta t} - \frac{gMm\delta t}{r(t_n)^2}\hat{\mathbf{r}}(t_n) \tag{16}$$

An important fact is, that the algorithm relates a new position with the two previous positions and the forces at the time, where the forces act. I.e., the momentum (or velocity) is not a dynamical variable in the discrete dynamics [1], and any expression for velocity, and thereby the kinetic energy is ad hoc.

The discrete time evolution with the constant time increment δt, obtained by Newton's central difference algorithm, starts from either two sets of positions, $\mathbf{r}(t_0), \mathbf{r}(t_0 - \delta t)$ (Verlet algorithm), or, as Newton illustrated, from a position $\mathbf{r}(t_0)$ and a previous change of position $\mathbf{r}(t_0) - \mathbf{r}(t_0 - \delta t) \equiv \delta t \mathbf{v}(t_0 - \delta t/2)$, in the time interval $t \in [t_0 - \delta t, t_0]$ (Leap frog or implicit Euler algorithm). The velocity $\mathbf{v}(t_n)$ at the time where the force acts, at the position $\mathbf{r}(t_n)$, is in general obtained by a central difference

$$\mathbf{v}(t_n) = \frac{\mathbf{v}(t_n + \delta t/2) + \mathbf{v}(t_n - \delta t/2)}{2} = \frac{\mathbf{r}(t_n + \delta t) - \mathbf{r}(t_n - \delta t)}{2\delta t}. \tag{17}$$

Newton's discrete time reversible dynamics has the same three invariances as his analytic dynamics. It conserves the (total) angular momentum, $\mathbf{L}$. The angular momentum, $\mathbf{L}(t_n)$ for a planet at the n'th time step (and using the Verlet-formulation, Equation 17 and the fact, that the force is in the direction of the discrete position) is

$$\begin{aligned}\frac{2\delta t}{m}\mathbf{L}(t_n) &= \mathbf{r}(t_n)\times(\mathbf{r}(t_{n+1})-\mathbf{r}(t_{n-1}))\\ &= \mathbf{r}(t_n)\times(2\mathbf{r}(t_n)-2\mathbf{r}(t_{n-1}))\\ &= \mathbf{r}(t_n-1)\times(\mathbf{r}(t_n)+\mathbf{r}(t_n)) =\\ = \mathbf{r}(t_n-1)\times(\mathbf{r}(t_n)-\mathbf{r}(t_{n-2})) &= \frac{2\delta t}{m}\mathbf{L}(t_{n-1}).\end{aligned} \tag{18}$$

It is straight forward to prove, that the constant area of the triangles in Newton's formulation of the discrete dynamics (Figure 1) is a consequence of the conserved angular momentum. Newton did not noticed in *Principia*, that the constant area of the triangles implies, that the discrete dynamics for a planet around the Sun obeys Kepler's second law, and that this law is a general law for two body discrete dynamics, independent of the nature of the central force.

If one determines the energy at the n'th time step by

$$E_{disc}(t_n) = \frac{1}{2}m\mathbf{v}(t_n)^2 - \frac{gMm}{r(t_n)}, \tag{19}$$

it fluctuates during the discrete time propagation, although the mean value remains constant.

C. The shadow Hamiltonian for the central difference algorithm

The points obtained by Newton's central difference algorithm for a simple harmonic force is located on a harmonic trajectory of a harmonic "shadow Hamiltonian" $\tilde{H}(\mathbf{q},\mathbf{p})$ [16], with position $\mathbf{q}$ and momentum $\mathbf{p}$ in the Lagrange-Hamilton equations. The shadow Hamiltonian $\tilde{H}$ for a symplectic and time-reversible discrete algorithm can in general be obtained from the corresponding $H(\mathbf{q},\mathbf{p})$ for the analytic dynamics by an asymptotic expansion in the time increment δt, if the potential energy is analytic [17–19],

$$\tilde{H} = H + \frac{\delta t^2}{2!}g(\mathbf{q},\mathbf{p}) + \mathcal{O}(\delta t^4), \tag{20}$$

The corresponding energy invariance, $\tilde{E}$, for the discrete dynamics in Cartesian coordinates for N particles is [16, 20, 21]

$$\tilde{E}_n = U(\mathbf{R}_n) + \frac{1}{2}m\mathbf{V}_n^2 + \frac{\delta t^2}{12}\mathbf{V}_n^T\mathbf{J}(\mathbf{R}_n)\mathbf{V}_n - \frac{\delta t^2}{24m}\mathbf{F}_n(\mathbf{R}_n)^2 + \mathcal{O}(\delta t^4), \tag{21}$$

where $\mathbf{J}$ is the Hessian, $\partial^2 U(\mathbf{q})/\partial \mathbf{q}^2$, of the potential energy function $U(\mathbf{q})$, the velocity of the N particles is $\mathbf{V}_n \equiv (\mathbf{v}_1, ..., \mathbf{v}_N)$, and the force with position $\mathbf{R} \equiv (\mathbf{r}_1, ..., \mathbf{r}_N)$ is $\mathbf{F}(\mathbf{R}) \equiv (\mathbf{f}_1(\mathbf{R}), ..., \mathbf{f}_N(\mathbf{R}))$.

The observed energy fluctuations for a complex system decreases by a factor of hundred or even more by including this correction in the expression for the energy and it indicates, that the expansion is rapidly converging for relevant time increments [21, 22].

The shadow energy at the n'th step for a planet, attracted by the Sun at a fixed position at the origin, can be obtained from the expressions in Appendix A in [21]. It is

$$\tilde{E}(t_n) = E_{disc}(t_n) - \frac{\delta t^2}{12}\left(\frac{3gMm}{r(t_n)^5}(\mathbf{v}(t_n)\mathbf{r}(t_n))^2 + \frac{gMm}{r(t_n)^3}\mathbf{v}(t_n)^2\right) - \frac{\delta t^2 (gMm)^2}{24 r(t_n)^4} + \mathcal{O}(\delta t^4). \quad (22)$$

IV. THE ORBIT OF A PLANET OBTAINED BY NEWTON'S DISCRETE ALGORITHM

The positions of a planet are obtained by Newton's central difference algorithm. The positions are determined by the time increment δt and by the same parameters as the analytic curve, e.g. gM, m, r_p and vy_p. The curves through the points are almost identical to the analytic ellipses, and the discrete dynamics obeys the same condition for a stable elliptic orbit as the analytic dynamics (Equation 15). Figure 2 shows the orbits, obtained with different start values of the velocity, vy_p [23].

The generation of positions by the central difference algorithm needs either two consecutive start positions, $\mathbf{r}(t_0 - \delta t)$ and $\mathbf{r}(t_0)$, or $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0 - \delta t/2)$. It is convenient to start the dynamics in perihelion (or aphelion) where $vx(t_0) = 0$. Due to the time reversibility of the discrete dynamics $vy(t_0 + \delta t/2) = vy(t_0 - \delta t/2)$ and $vx(t_0 + \delta t/2) = -vx(t_0 - \delta t/2)$ at perihelion. The first discrete position away from the perihelion, $x(t_0 + \delta t), y(t_0 + \delta t)$, is

$$\mathbf{r}(t_0 + \delta t) = x(t_0 + \delta t), y(t_0 + \delta t) = -r_p + \frac{1}{2}\frac{gM\delta t^2}{r_p^2}, \delta t v y_p, \quad (23)$$

and since $x(t_0 + \delta t) = x(t_0 - \delta t)$ due to the time symmetry, the discrete dynamics starts with an energy $E_{disc}(t_0)$ at time t_0=0, which is equal to the constant energy E in the analytic dynamics.

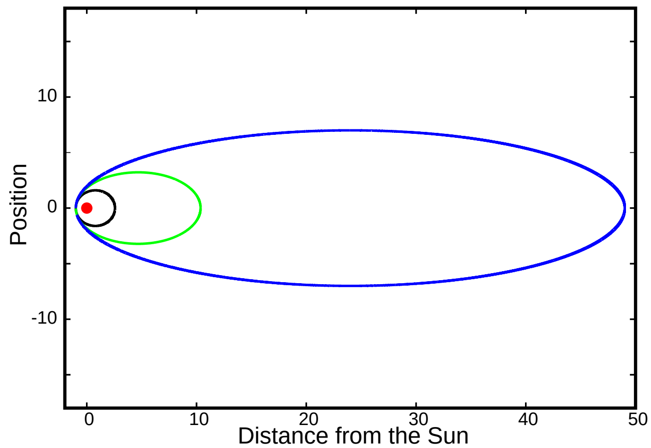

FIG. 2. The orbits of an Earth-like planet. The curves are obtained by Newton's central difference algorithm from the perihelion at $\mathbf{r}(t_0) = (-1, 0)$ with $gM = m = 1$, and with different velocities vy_p. The red filled circle is the position of the Sun and the three curves are for vy_p= 1.2 (black); 1.3 (green) and 1.4(blue), respectively.

A. A shadow Hamiltonian and the functional form of the orbits for the discrete dynamics

The question is: Is there a shadow Hamiltonian for the discrete dynamics of a planet's orbital motion, and if so, what is the functional form of the analytic function for $\tilde{H}$. Since the discrete dynamics for δt going to zero converges to the analytic dynamics with elliptic motion, it is natural to fit an ellipse to the discrete points.

The main investigation is for an Earth-like planet with $vy_p = 1.2$ at $r_p = -1$ and with $gM = m = 1$. The results are given in Table I. with data for different values of $n = T(orbit)/\delta t$, where $T(orbit)$ is the orbit time with analytic dynamics (Equation 13). The investigation shows several things.

The discrete points are with high precision on an ellipse. Figure 3 shows the planet's positions near perihelion and when the position is updated every $\delta t = T(orbit)/365$, or $\approx$ 24 hours for an Earth-like planet. Column 2 and 3 in the Table give the fitted values for the axes and with the rms stand deviations of the fits in column 4. E. g. a deviation of $3. \times 10^{-8}$ corresponds to $\approx$ 3-4 km in the case of planet Earth.

The mean energies, $< E_{disc} >$ and $< \tilde{E}_{disc} >$ are given in column 5 and 6. The observed energy fluctuations are decreased by a factor of the order $\approx 10^3$ to 10^5 just by inclusion of the first order correction (Equation 22). Figure 4 shows the energy evolution during tree

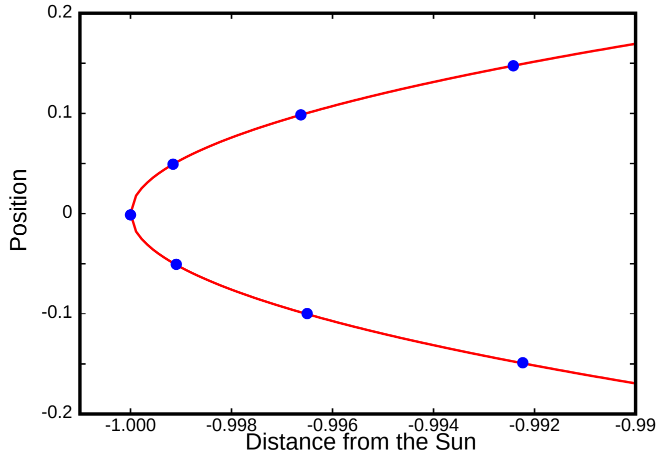

FIG. 3. The discrete positions of an Earth-like planet near perihelion. The discrete positions (red filled circles) are obtained by Newton's central difference algorithm with $\delta t = T(orbit)/365$, i.e. for an Earth-like planet every 24 hours. The full line is an ellipse determined from the 365 discrete points by fitting the axes of an ellipse.

times in the orbit. The tiny energy variations of the shadow energies are shown in the insert.

The discrete dynamics was obtained for other values of $vy_p, gM.m$ and r_p and confirmed the result, that the discrete dynamics behaves as the analytic. The discrete positions were located on ellipses and the energies, $\tilde{E}(t_n)$ were almost constant by inclusion of the first order term (Equation 22) in $E_{disc}(t_n)$.

Table 1. Principial axis and discrete energies for $r_p = -1$, $vy_p = 1.2$, $gMm/r_p = -1$ and $\delta t = T(orbit)/n$.

n	Major axis	Minor axis	rms	E_{disc}	$\tilde{E}$
365	1.7867062	1.60399	$4.\times10^{-4}$	-0.27988$\pm3.10^{-5}$	-0.2798678 $\pm1.10^{-8}$
10^3	1.7858364	1.603624	$2.\ \times10^{-4}$	-0.279984 $\pm3.10^{-6}$	-0.2799823897 $\pm3.^{-10}$
10^4	1.7857156	1.603568016	$2.\ \times10^{-7}$	-0.27999984 $\pm4.10^{-8}$	-0.2799998239055 $\pm7.10^{-13}$
10^5	1.78571423	1.603567457	$3.\ \times10^{-8}$	-0.2799999985 $\pm4.10^{-10}$	-0.27999999823913 $\pm1.10^{-14}$
∞	1.78571429	1.603567451	0	-0.28	-0.28

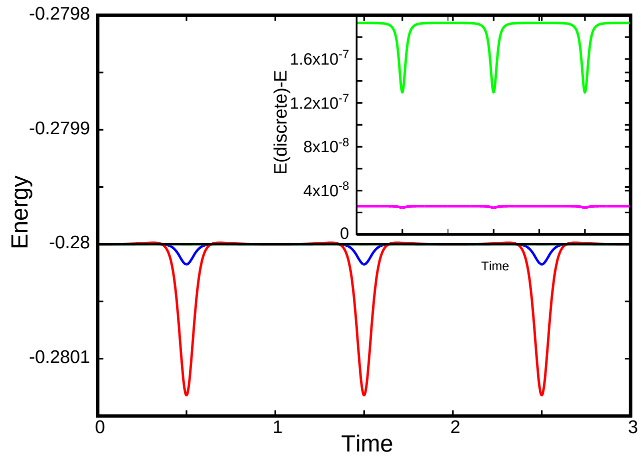

FIG. 4. The energies $E(t_n)$ and $\tilde{E}(t_n)$ for for the circulation of a planet tree times in its elliptic orbit. The discrete values are obtained by starting from the aphelion $\mathbf{r}(t_0) = (r_{max}, 0)$ with r_{max} and vy_{min} obtained from $r_p = -1$, $vy_p = 1.2$, $gM = m = 1$ and $\delta t_1 = T(orbit)/365$ and $\delta t_2 = T(orbit)/1000$ ($\approx$ one day and eight hours, respectively). Red: $E(t_n)$ with δt_1; blue: $E(t_n)$ with δt_2; black: $E(analytic) = -0.28$. The inset shows the small energy differences between the corresponding shadow energies, $\Delta E(t_n) = \tilde{E}(t_n) - E(analytic)$. Green: $\Delta E(t_n)$ for δt_1; Magenta: $\Delta E(t_n)$ for δt_2.

The existence of a shadow Hamiltonian for the discrete dynamics of a celestial body implies that the positions, obtained by the discrete dynamics, are the exact solutions to a discrete classical dynamics with the same dynamics invariances as the analytic dynamics: conservation of momenta, angular momenta and total energy.

V. DISCUSSION

Isaac Newton and Robert Hooke used the geometric implementation (Figure 1) of the central difference algorithm to obtain orbital moves [24]; but they were of course not aware of, that the algorithm has the same qualities as Newton's analytic dynamics. But despite the same dynamic invariances, the time reversibility and symplectic behaviour there is, however, one fundamental difference between the two dynamics. Only the positions and time are variables, the momenta are just constructed expressions, obtained from the positions.

The discrete dynamics has the same qualitative behaviour and the same invariances as Newtonian dynamics, and it raises the question: Which of these two formulations are correct, or alternatively, the most appropriate formulation of classic dynamics? In this context Noble laureate T. D. Lee wrote that he "wish to explore an alternative point of view: that physics should be formulated in terms of difference equations and that these difference equations could exhibit all the desirable symmetry properties and conservation laws". Lee's analysis covers not only classical mechanics [1], but also non relativistic quantum mechanics and relativistic quantum field theory [2], and Gauge theory and Lattice Gravity [3]. The discrete dynamics is obtained by treating positions and time, *but not momenta*, as a discrete dynamical variables.

T. D. Lee wrote in his concluding remarks[1] : *For more than three centuries we have been influenced by the precept that fundamental laws of physics should be expressed in terms of differential equations. Difference equations are always regarded as approximations. Here I try to explore the opposite: Difference equations are more fundamental, and differential equations are regarded as approximation.*

If so, Isaac Newton will also be the founder of discrete dynamics.

VI. ACKNOWLEDGMENT

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